On the In-and-Circumscribed Triangles of the Plane Rational Quartic Curve

By

Joseph Nelson Rice

A DISSERTATION

Submitted to the Faculty of Sciences of the Catholic University of America in partial fulfilment of the requirements for the degree of Doctor of Philosophy.

Washington, D. C.
June, 1917.

On the In-and-Circumscribed Triangles of the Plane Rational Quartic Curve

By

Joseph Nelson Rice

A DISSERTATION

Submitted to the Faculty of Sciences of the Catholic University of America in partial fulfilment of the requirements for the degree of Doctor of Philosophy.

Washington, D. C.
June, 1917.

NATIONAL CAPITAL PRESS, INC., WASHINGTON, D. C.

ON THE IN-AND-CIRCUMSCRIBED TRIANGLES OF THE PLANE RATIONAL QUARTIC CURVE

INTRODUCTION

The question of the existence of simultaneously inscribed and circumscribed triangles has received considerable attention, most of which, however, has been directed to the consideration of poristic cases.

R. A. Roberts[1] investigates the possibility of the existence of an infinite number of closed polygons simultaneously inscribed in, and circumscribed about, a unicursal quartic. In this connection he discusses only the case of the nodo-bicuspidal quartic, and shows that, for the triangle, the results are irrelevant.

Morley,[2] in his article entitled "The Poncelet Polygons of a Limacon," shows that the results obtained are poristic, except for the triangle, in which case they are irrelevant.

Cayley,[3] in "On a Triangle In-and-circumscribed about a Quartic Curve," shows that the binodal quartic

$$(x^2-1)^2+\left(y^2-\frac{1}{2}\right)^2=1$$

has four such triangles. These are such that two of the sides are tangent to the inner loop and the third to the outer.

The same author, in "On the Problem of the In-and-circumscribed Triangle," considers the following problem: "required the number of triangles the angles of which are situate in a given curve or curves, and the sides of which touch a given curve or curves."[5] He discusses 52 cases of the problem, according as the curves containing the angles or touching the sides are, or are not, distinct curves. The simplest case is when all the curves are

---

[1] *Proceedings of the London Mathematical Society*, Vol. xvi, p. 53.
[2] *Ibid.*, Vol. xxix, pp. 83–97.
[3] Collected Mathematical Works, Vol. v, pp. 489–492.
Philosophical Transactions, t. XXX (1865), pp. 340–342.
[4] Collected Mathematical Works, Vol. viii, pp. 212–258.
Philosophical Transactions. t. clxi (1871), pp. 369–412.
[5] Collected Mathematical Works, Vol. viii, p. 567.
Reports of the British Associations for the Advancement of Science, 1865–1873. *Report*, 1870, pp. 9–10.

distinct; the number of triangles is here equal to $2aceBDF$, where a, c, e are the orders of the curves containing the angles respectively; and B, D, F are the classes of the curves touched by the sides respectively. The last and most difficult case is when the six curves are all of them one and the same curve. The number of triangles is here equal to one-sixth of

$$\begin{aligned}&2A^3a^3-18(A^3a^2+A^2a^3)+(A^4+a^4)+52(A^3a+Aa^3)+162A^2a^2\\&-46(A^3+a^3)-420(A^2a+Aa^2)+221(A^2+a^2)+704Aa\\&+172(A+a)+\alpha[-9(A^2+a^2)-12Aa+135(A+a)-600]\end{aligned}$$

where a is the order, A the class of the curve; α is the number, three times the class+number of cusps, or, what is the same, three times the order+the number of inflexions.

It is to be noted that this formula gives the same number of triangles as has been found by the method used later. For example, in the case of the rational quartic, where $a=4$, $A=6$, $\alpha=18$, the number of triangles is 8, which corresponds to that found on page 18. For the cuspidal quartic, where $a=4$, $A=5$, $\alpha=16$, the number is two, which also corresponds to the number found on page 22.

In this paper it is proposed to look into the existence and actual number of such triangles for the following types of rational quartics:

I. Quartic with three double points.
II. Quartic with one double point and a tacnode.
III. Quartic with a triple point.
IV. Quartic with two double points and a cusp.

This discussion was led up to by preliminary work on the three-cusped rational quintic. Upon subjection to a quadratic transformation this curve goes into a rational quartic, which, it will be shown, has triangles of the kind here mentioned. Accordingly, it will first be proved that the quintic can have certain conditions imposed upon its coefficients so that it may acquire an additional cusp or a tacnode without degenerating. It will also be shown that it cannot have a triple point.

THE THREE-CUSPED RATIONAL QUINTIC

This quintic may be expressed parametrically as follows:

$$x_i = \frac{t-\alpha_i}{(t-\beta_i)^2}\ , \ (i=1,\ 2,\ 3)$$

that is,

$$x_1 = (t-\alpha_1)(t-\beta_2)^2\ (t-\beta_3)^2$$
$$x_2 = (t-\alpha_2)(t-\beta_3)^2\ (t-\beta_1)^2$$
$$x_3 = (t-\alpha_3)(t-\beta_1)^2\ (t-\beta_2)^2$$

This quintic has cusps at the vertices of the fundamental triangle, the values of the parameters thereat being

$$t = \beta_i, \quad (i=1,\ 2,\ 3)$$

as is readily seen by considering the common intersections on any two of the lines

$$x_i = 0, \quad (i=1,\ 2,\ 3)$$

Consider, for example, the intersections on $x_1=0$ and $x_2=0$. They have in common β_3 taken twice. These common points must be at the intersection of the lines themselves, viz., at the vertex three.

For the purpose of more ready computation, specialize this quintic by letting the values of the parameters at the cusps be $0,\ 1,\ \infty$. The parametric equations of the curve may then be written as:

$$x_1 = \frac{t-\alpha}{t^2} \qquad\qquad x_1 = (t-\alpha)(t-1)^2$$

$$x_2 = \frac{t-\beta}{(t-1)^2} \qquad \text{or} \qquad x_2 = (t-\beta)t^2$$

$$x_3 = t-\gamma \qquad\qquad x_3 = (t-\gamma)(t-1)^2 t^2$$

The Plücker numbers of the curve are:

$$m=5, \qquad \delta=3, \qquad \varkappa=3$$
$$n=5, \qquad \tau=3, \qquad \iota=3$$

It is proposed to derive the conditions, which must be imposed upon the parameters α, β, γ, so that the quintic may have,

I. A fourth cusp,
II. A tacnode,
III. A triple point.

It will be seen that, excluding the case of the triple point, the quintic can have such extra points in addition to the three cusps

at the vertices of the fundamental triangle. As preliminary, there will be derived:

(1) The cubic equation connecting the three parameters at the three flexes;

(2) The sextic equation connecting the six parameters at the three double points.

(1) The Cubic whose Three Roots are the Flexes.

Let $Lx_1+Mx_2+Nx_3=0$ be a flex-tangent.

Then $\quad Lf_1(t)+Mf_2(t)+Nf_3(t)=0$ has a triple root,

and $\quad Lf_1'(t)+Mf_2'(t)+Nf_3'(t)=0$ has the same root doubled,

also $\quad Lf_1''(t)+Mf_2''(t)+Nf_3''(t)=0$ has the same root once.

From these three equations eliminate L, M, N, then

$$\begin{vmatrix} f_1(t) & f_2(t) & f_3(t) \\ f_1'(t) & f_2'(t) & f_3'(t) \\ f_1''(t) & f_2''(t) & f_3''(t) \end{vmatrix} = 0, \text{ is the cubic of flexes.}$$

For the functions of t and their derivatives substitute their corresponding values and reduce, thus giving as the required equation:

$$3(\beta-\alpha-2)t^3+(12\alpha+4\beta+2\gamma+\alpha\gamma-\beta\gamma)t^2 \\ +(\gamma-\beta-12\alpha\beta-4\alpha\gamma-2\beta\gamma)t+3\alpha(\beta-\gamma+2\beta\gamma)=0 \quad \ldots (1)$$

If one of these flexes coincides with one of the three cusps, that is, if t has any one of the values 0, 1, ∞, there is then at this point a cusp of the second kind.* Hence, the condition that there be a cusp of this character at:

$$\begin{array}{lll} t=0 & \text{is} & \beta-\gamma+2\beta\gamma=0 \\ t=1 & .. & 3\alpha+\gamma-2\alpha\gamma-2=0 \\ t=\infty & .. & \alpha-\beta+2=0 \end{array}$$

(2) The Sextic connecting the Six Parameters at the Double Points.

Let λ and μ be the parameters at one of the double points.

Then
$$\frac{\lambda-\alpha}{\lambda^2} : \frac{\lambda-\beta}{(\lambda-1)^2} : \lambda-\gamma = \frac{\mu-\alpha}{\mu^2} : \frac{\mu-\beta}{(\mu-1)^2} : \mu-\gamma$$

$$\frac{\lambda-\alpha}{\lambda^2} \cdot (\mu-\gamma) = \frac{\mu-\alpha}{\mu^2} \cdot (\lambda-\gamma)$$

---

* Salmon: Higher Plane Curves (French Edition), Chap. ii, p. 70.

Cayley: "On the Cusp of the Second Kind or Nodecusp;" *Collected Mathematical Works*, Vol. v, pp. 265, 266.

Quarterly Journal of Pure and Applied Mathematics, Vol. vi (1864), pp. 74, 75.

These equations reduce to

$$(\lambda-\alpha)\mu^2+(\lambda-\alpha)(\lambda-\gamma)\mu-\alpha\lambda(\lambda-\gamma)=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$
$$(\lambda-\beta)\mu^2+(\lambda-\beta)(\lambda-\gamma-2)\mu-(\beta\lambda^2-\beta\gamma\lambda-2\beta\lambda+2\beta\gamma+\beta-\gamma)=0\quad(2)$$

Eliminate μ from these equations. The eliminant is

$$\begin{vmatrix} (\lambda-\alpha)(\lambda-\gamma) & (\lambda-\beta)(\lambda-\gamma-2) \\ (\lambda-\alpha) & (\lambda-\beta) \end{vmatrix}\times$$
$$\begin{vmatrix} (\lambda-\alpha)(\lambda-\gamma) & -\alpha\lambda(\lambda-\gamma) \\ (\lambda-\beta)(\lambda-\gamma-2) & -(\beta\lambda^2-\beta\gamma\lambda-2\beta\lambda+2\beta\gamma+\beta-\gamma) \end{vmatrix}$$
$$+\begin{vmatrix} -\alpha\lambda(\lambda-\gamma) & -(\beta\lambda^2-\beta\gamma\lambda-2\beta\lambda+2\beta\gamma+\beta-\gamma) \\ (\lambda-\alpha) & (\lambda-\beta) \end{vmatrix}^2=0$$

On expanding this gives

$$\begin{aligned}&(\alpha-\beta)(\alpha-\beta+2)\lambda^6-2(\alpha-\beta)(\alpha-\beta+2)(1+\gamma)\,\lambda^5\\
&\quad+\{2(\alpha-\beta)(\gamma^2+\alpha\gamma-\alpha\beta+2\alpha+\beta+3\gamma)-2(2\beta\gamma+\beta-\gamma)\\
&\qquad+(\alpha\gamma-\beta\gamma-2\beta)^2\}\,\lambda^4\\
&\qquad-2\{(\alpha+\beta)(\beta\gamma^2+\alpha\gamma^2+\alpha\beta+2\beta\gamma+3\alpha\gamma)-2(\alpha+\beta+\gamma)\\
&\qquad(2\beta\gamma+\beta-\gamma)-(\alpha\gamma-\beta\gamma-2\beta)(2\beta\gamma+2\alpha\beta+\beta-\gamma)\}\,\lambda^3\\
&-\{2(2\beta\gamma+\beta-\gamma)(\alpha^2\gamma-\alpha\beta\gamma+\alpha^2+2\alpha\gamma+\beta\gamma)+2\alpha\beta\gamma(\alpha-\beta)(\gamma+2)\\
&\qquad-(2\beta\gamma+2\alpha\beta+\beta-\gamma)^2\}\,\lambda^2\\
&+2\alpha(2\beta\gamma+\beta-\gamma)(\gamma-\beta)(\alpha+1)\lambda+\alpha^2(\beta-\gamma)(2\beta\gamma+\beta-\gamma)=0\ldots(3)\end{aligned}$$

It may be noted here that, according to Cayley's analysis,* a cusp of the second kind takes up a double point. If $\alpha-\beta+2=0$, then the sextic has $t=\infty$ for a double root. But this is the condition that the quintic have a node-cusp at $t=\infty$.

Equations (1) and (2) may be written as follows:

$$\sigma_1\sigma_2-\alpha(\sigma_1^2-\sigma_2)-\gamma\sigma_2+\alpha\gamma\sigma_2=0\ldots\ldots\ldots\ldots(4)$$
$$\sigma_1\sigma_2-\beta(\sigma_1^2-\sigma_2)-(\gamma+2)\sigma_2+(\beta\gamma+2\beta)\sigma_1-(\beta-\gamma+2\beta\gamma)=0..(5)$$

where $\sigma_1=\lambda+\mu$ and $\sigma_2=\lambda\mu$.

Eliminating σ_2 from (4) and (5), the result is:

$$\begin{aligned}&(\alpha-\beta)\sigma_1^3-2(\alpha-\beta)(\gamma+1)\sigma_1^2+\gamma^2(\alpha-\beta)+2\alpha(\beta+\gamma)\\
&\qquad-(\beta-\gamma+4\beta\gamma)\sigma_1-(\alpha-\gamma)(\beta-\gamma+2\beta\gamma)=0\ldots..(6)\end{aligned}$$

I. The Condition for a Fourth Cusp

It has been seen that this quintic has cusps at the three vertices of the fundamental triangle, the values of the parameters at these points being 0, 1, ∞. If a fourth cusp be possible, let the value of the parameter thereat be $t=\tau$.

---

* Cayley: "On the Cusp of the Second Kind of Node-Cusp;" *Collected Mathematical Works*, Vol. v, pp. 265, 266.

Join 1τ. The equation giving the parameters on the join is

$$\begin{vmatrix} \frac{t-\alpha}{t^2} & \frac{t-\beta}{(t-1)^2} & t-\gamma \\ \frac{\tau-\alpha}{\tau^2} & \frac{\tau-\beta}{(\tau-1)^2} & \tau-\gamma \\ 1 & 0 & 0 \end{vmatrix} = 0$$

Throwing out the factor $\frac{t-\tau}{(\tau-1)^2}$, putting $t=\tau$, and reducing, this becomes:

$$2\tau^2-(3\beta+\gamma)\tau+(\beta-\gamma+2\beta\gamma)=0 \ldots\ldots\ldots\ldots (1)$$

Similarly, the equation giving the parameters on the join of 2τ is,

$$\begin{vmatrix} \frac{t-\alpha}{t^2} & \frac{t-\beta}{(t-1)^2} & t-\gamma \\ \frac{\tau-\alpha}{\tau^2} & \frac{\tau-\beta}{(\tau-1)^2} & \tau-\gamma \\ 0 & 1 & 0 \end{vmatrix} = 0$$

which reduces to:

$$2\tau^2-(3\alpha+\gamma)\tau+2\alpha\gamma=0 \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Eliminate τ from (1) and (2). The eliminant is,

$$\begin{vmatrix} 2 & -3\beta-\gamma & \beta-\gamma+2\beta\gamma & 0 \\ 0 & 2 & -3\beta-\gamma & \beta-\gamma+2\beta\gamma \\ 2 & -3\alpha-\gamma & 2\alpha\gamma & 0 \\ 0 & 2 & -3\alpha-\gamma & 2\alpha\gamma \end{vmatrix} = 0$$

which reduces to:

$$2\gamma^2(\alpha-\beta)^2-(5\gamma-9\alpha)(\alpha-\beta)(\beta-\gamma)+2(\beta-\gamma)^2=0$$

which is the required condition.

A cusp may result by the coming together of two neighboring inflexions as is readily seen by considering a penultimate case. Hence the condition for a fourth cusp should be contained, as a factor, in the discriminant of the equation of the flex cubic (Equation I, p. 6).

This was not shown in general, as the work involved considerable algebraic difficulty. In the special cases attempted, it was, however, proven to be true, as the following example will show.

As previously shown, the condition for a cusp of the second kind at $t=\infty$, is:

$$\alpha-\beta+2=0$$

The discriminant of the flex-cubic is then the eliminant of

$$16(2\alpha+1)t-(2+3\gamma+25\alpha+6\alpha\gamma+12\alpha^2)=0$$

$$8(2\alpha+1)t^2-2(2+3\gamma+25\alpha+6\alpha\gamma+12\alpha^2)t+9\alpha(2+\alpha+3\gamma+2\alpha\gamma)=0$$

Therefore
$$
\begin{aligned}
E &= (2+3\gamma+25\alpha+6\alpha\gamma+12\alpha^2)^2 \\
&\quad -96\alpha(2\alpha+1)(2+\alpha+3\gamma+2\alpha\gamma) \\
&= 144\alpha^4-240\alpha^3\gamma+36\alpha^2\gamma^2+408\alpha^3-396\alpha^2\gamma+36\alpha\gamma^2 \\
&\quad -193\alpha^2-114\alpha\gamma+9\gamma^2+12\gamma+4=0
\end{aligned}
$$

The condition for a fourth cusp reduces to

$$4\alpha^2-6\alpha\gamma+7\alpha-3\gamma-2=0$$

This divides into E, giving for quotient,

$$36\alpha^2-6\alpha\gamma+39\alpha-3\gamma-2=0$$

II. The Condition for a Tacnode

A tacnode may arise by the coming together of two double points. The joins of the three double points, two and two, intersect the quintic in an additional fifth point each. If two of the double points coincide, then two of these additional points will also coincide. It is necessary to find the cubic equation connecting these three points.

Let
$$
\begin{aligned}
&A_1x_1+A_2x_2+A_3x_3 \\
&= A_1(t-\alpha)(t-1)^2+A_2(t-\beta)t^2+A_3(t-\gamma)(t-1)^2t^2 \\
&= A_3t^5-(2+\gamma)A_3t^4+\{A_3(1+2\gamma)+A_2+A_1\}t^3-\{A^3\gamma+A_2\beta \\
&\qquad +A_1(2+\alpha)\}\, t^2+A_1(1+2\alpha)t-A_1\alpha=0,
\end{aligned}
$$

be the equation of the five parameters of the points of intersection of any line with the quintic.

Then $s_1=2+\gamma$ is the sum of these five parameters,

and $\quad \dfrac{s_4}{s_5}=\dfrac{1+2\alpha}{\alpha}$ is the sum of their reciprocals.

From the sextic of the parameters of the double points, it is seen that $s_1'=2(1+\gamma)$ is the sum of the six parameters thereat,

and $\quad \dfrac{s_5'}{s_6'}=\dfrac{2(\alpha+1)}{\alpha}$ is the sum of their reciprocals.

It is readily seen that, if S_1, S_2, S_3 be the symmetric functions of the three extra points of intersection with the curve of the joins of the double points, two and two, then,

$$S_1=3s_1-2s_1'=3(2+\gamma)-4(1+\gamma)=2-\gamma$$

and
$$\frac{S_2}{S_3}=3\frac{s_4}{s_5}-2\frac{s_5'}{s_6'}=\frac{3+6\alpha}{\alpha}-\frac{4\alpha+4}{\alpha}=\frac{2\alpha-1}{\alpha}$$

To determine S_3:

Let the parameters at the double points be t_1 and t_2, t_3 and t_4, t_5 and t_6. Then the remaining parameter on the

$$\begin{array}{ll} \text{join of } (t_1, t_2) \text{ and } (t_3, t_4) \text{ is} & (2+\gamma)-(t_1+t_2+t_3+t_4) \\ \text{join of } (t_1, t_2) \text{ and } (t_5, t_6) \text{ is} & (2+\gamma)-(t_1+t_2+t_5+t_6) \\ \text{join of } (t_3, t_4) \text{ and } (t_5, t_6) \text{ is} & (2+\gamma)-(t_3+t_4+t_5+t_6) \end{array}$$

$$S_1=3(2+\gamma)-2\Sigma t_i=3(2+\gamma)-4(1+\gamma)=2-\gamma.$$

$$\begin{aligned} S_3 &= \{(2+\gamma)-(t_1+t_2+t_3+t_4)\}\{(2+\gamma)-(t_1+t_2+t_5+t_6)\}\{(2+\gamma) \\ &\quad -(t_3+t_4+t_5+t_6)\} \\ &= (2+\gamma)^3-(2+\gamma)^2(2\Sigma t_i)+(2+\gamma)[(\Sigma t_i)^2+(t_1+t_2)(t_3+t_4) \\ &\quad +(t_1+t_2)(t_5+t_6)+(t_3+t_4)(t_5+t_6)]-\Sigma t_i\{(t_1+t_2)(t_3+t_4) \\ &\quad +(t_1+t_2)(t_5+t_6)+(t_3+t_4)(t_5+t_6)\}+(t_1+t_2)(t_3+t_4)(t_5+t_6) \end{aligned}$$

From equation (6), page 7,

$$\Sigma t_i=2(\gamma+1)$$

$$(t_1+t_2)(t_3+t_4)+(t_1+t_2)(t_5+t_6)+(t_3+t_4)(t_5+t_6) = \frac{\gamma^2(\alpha-\beta)+2\alpha(\beta+\gamma)-(\beta-\gamma+4\beta\gamma)}{\alpha-\beta}$$

$$(t_1+t_2)(t_3+t_4)(t_5+t_6)=\frac{(\alpha-\gamma)(\beta-\gamma+2\beta\gamma)}{\alpha-\beta}$$

Substitute these values in S_3 and reduce:

$$S_3=\frac{\alpha(\beta-\gamma)}{\alpha-\beta}$$

But
$$\frac{S_2}{S_3}=\frac{2\alpha-1}{\alpha}$$

Therefore
$$S_2=\frac{(2\alpha-1)(\beta-\gamma)}{\alpha-\beta}.$$

The cubic equation connecting the parameters of the three required points of intersection is:

$$\lambda^3-S_1\lambda^2+S_2\lambda-S_3=0$$

$$\lambda^3-(2-\gamma)\lambda^2+\frac{(2\alpha-1)(\beta-\gamma)}{\alpha-\beta}\lambda-\frac{\alpha(\beta-\gamma)}{\alpha-\beta}=0$$

$$(\alpha-\beta)\lambda^3-(\alpha-\beta)(2-\gamma)\lambda^2+(2\alpha-1)(\beta-\gamma)\lambda-\alpha(\beta-\gamma)=0$$

If this cubic have a double root, it means that two of the double roots coincide, thus giving a tacnode. The cubic will have a double root if its discriminant is zero. The discriminant is the eliminant of

$$3(\alpha-\beta)\lambda^2+2(\alpha-\beta)(\gamma-2)\lambda+(2\alpha-1)(\beta-\gamma)=0$$

$$(\alpha-\beta)(\gamma-2)\lambda^2+2(2\alpha-1)(\beta-\gamma)\lambda-3\alpha(\beta-\gamma)=0$$

$$E=\begin{vmatrix} 3(\alpha-\beta) & 2(\alpha-\beta)(\gamma-2) & (2\alpha-1)(\beta-\gamma) & 0 \\ 0 & 3(\alpha-\beta) & 2(\alpha-\beta)(\gamma-2) & (2\alpha-1)(\beta-\gamma) \\ (\alpha-\beta)(\gamma-2) & 2(2\alpha-1)(\beta-\gamma) & -3\alpha(\beta-\gamma) & 0 \\ 0 & (\alpha-\beta)(\gamma-2) & 2(2\alpha-1)(\beta-\gamma) & -3\alpha(\beta-\gamma) \end{vmatrix}$$

$$\begin{aligned}&=3(\alpha-\beta)(\beta-\gamma)(\alpha-\gamma)[4\alpha\beta\gamma^2(\beta-\alpha)+4\alpha(8\alpha\beta^2+\alpha\gamma^2-3\alpha\beta\gamma-\beta\gamma^2\\&\quad-5\beta^2\gamma)+(36\alpha\beta\gamma-61\alpha^2\beta+13\alpha^2\gamma+13\alpha\beta^2-\alpha\gamma^2+\beta\gamma^2-\beta^2\gamma)\\&\quad+4(8\alpha^2-5\alpha\gamma-3\alpha\beta-\beta\gamma+\beta^2)+4(\gamma-\beta)]=0\end{aligned}$$

$\alpha=\beta$, $\beta=\gamma$, $\gamma=\alpha$ are the conditions that a branch of the curve passes through one of the cusps. Since the fifth intersection on the sides of the fundamental triangle is given by $t=\alpha$, β, or γ, if two of these are equal, then the points of intersection must be at points common to the sides of the triangle, viz., the vertices, at which are the cusps. The remaining factor then must be the condition that the quintic acquire a tacnode.

The Triple Point

If the quintic can have a triple point, let the value of the parameter thereat be $t=\tau$.

The join of 1τ is $\begin{vmatrix} \dfrac{t-\alpha}{t^2} & \dfrac{t-\beta}{(t-1)^2} & t-\gamma \\ \dfrac{\tau-\alpha}{\tau^2} & \dfrac{\tau-\beta}{(\tau-1)^2} & \tau-\gamma \\ 1 & 0 & 0 \end{vmatrix}=0$

which reduces to

$$\begin{aligned}&(\tau-\beta)t^2+(\tau-\beta)(\tau-\gamma-2)t+\\&\{-\beta\tau^2+\beta(\gamma+2)\tau-(\beta-\gamma+2\beta\gamma)\}=0\ldots\ldots\ldots\ldots(1)\end{aligned}$$

The join of 2τ is $\begin{vmatrix} \dfrac{t-\alpha}{t^2} & \dfrac{t-\beta}{(t-1)^2} & t-\gamma \\ \dfrac{\tau-\alpha}{\tau^2} & \dfrac{\tau-\beta}{(\tau-1)^2} & \tau-\gamma \\ 0 & 1 & 0 \end{vmatrix}=0$

which reduces to

$$(\tau-\alpha)t^2+(\tau-\gamma)(\tau-\alpha)t-\alpha\tau(\tau-\gamma)=0\ldots\ldots\ldots\ldots(2)$$

These two equations give the remaining points of intersection of 1τ and 2τ with the curve. For a triple point these must coincide. Hence

$$\begin{aligned}&(\tau-\beta):(\tau-\alpha)(\tau-\gamma-2):-\beta\tau^2+\beta(\gamma+2)\tau-(\beta-\gamma+2\beta\gamma)\\&\qquad=(\tau-\alpha):(\tau-\alpha)(\tau-\gamma):-\alpha\tau(\tau-\gamma)\end{aligned}$$

Hence

$$(\tau-\beta):-\beta\tau^2+\beta(\gamma+2)\tau-(\beta-\gamma+2\beta\gamma)=(\tau-\alpha):-\alpha\tau(\tau-\gamma)$$

and

$$(\tau-\beta)(\tau-\gamma-2):-\beta\tau^2+\beta(\gamma+2)\tau-(\beta-\gamma+2\beta\gamma)$$
$$=(\tau-\alpha)(\tau-\gamma):-\alpha\tau(\tau-\gamma)$$

That is,

$$(\alpha-\beta)\tau^3+(\beta\gamma-\alpha\gamma+2\beta)\tau^2-(2\alpha\beta+2\beta\gamma+\beta-\gamma)\tau$$
$$+\alpha(\beta-\gamma+2\beta\gamma)=0 \ldots\ldots \quad (3)$$

and

$$(\alpha-\beta)\tau^3+(\alpha-\beta)(\gamma+2)\tau^2-(\beta-\gamma+2\beta\gamma)\tau+\alpha(\beta-\gamma+2\beta\gamma)=0 \ldots (4)$$

Eliminate τ from (3) and (4)

$$E=(\beta-1)^2(\beta-\gamma)(\alpha-\beta)=0$$

As previously shown, $\beta=\gamma$, $\alpha=\beta$, are the conditions that a branch of the curve passes through one of the cusps. If $\beta=1$, then the quintic degenerates, that is, there can be no triple point.

A second proof of the non-existence of the triple point will now be given.

Subject the quintic to the quadratic transformation $y_i=\dfrac{1}{x_i}$.

The curve becomes a quartic with the following parametric equations:

$$y_1=t^2(t-\beta)(t-\gamma)=t^4-(\beta+\gamma)t^3+\beta\gamma t^2$$
$$y_2=(t-1)^2(t-\alpha)(t-\gamma)=t^4-(\alpha+\gamma+2)t^3+\{\alpha\gamma+2(\alpha+\gamma)+1\}t^2$$
$$-(\alpha+\gamma+2\alpha\gamma)t+\alpha\gamma$$
$$y_3=(t-\alpha)(t-\beta)=t^2-(\alpha+\beta)t+\alpha\beta$$

From the five column matrix formed by the coefficients of the above three equations, the following ten determinants are derived:

$$24\Delta_{012}=-(\alpha-\beta+2)$$
$$16\Delta_{013}=(\alpha+\beta)(\alpha-\beta+2)$$
$$4\Delta_{014}=\alpha\beta(\alpha-\beta+2)$$
$$24\Delta_{023}=-(\alpha+\beta)(\alpha\gamma-\beta\gamma+2\alpha)-(\beta-\gamma+2\beta\gamma)$$
$$6\Delta_{024}=\alpha[\beta(\alpha\gamma-\beta\gamma+2\alpha)+(\beta-\gamma+2\beta\gamma)]$$
$$4\Delta_{034}=-\alpha^2(\beta-\gamma+2\beta\gamma)$$
$$96\Delta_{123}=\gamma^2(\alpha+\beta)(\alpha-\beta+2)+(\beta+\gamma)[2\alpha^2+(2\alpha+1)(\beta-\gamma)]$$
$$24\Delta_{124}=-\alpha\beta\gamma^2(\alpha-\beta+2)-\alpha(\beta+\gamma)(\beta-\gamma+2\alpha\beta)$$
$$16\Delta_{134}=\alpha^2(\beta+\gamma)(\beta-\gamma+2\alpha\beta)$$
$$24\Delta_{234}=-\alpha^2\beta\gamma(\beta-\gamma+2\beta\gamma)$$

If
$$a_0-a_1t+a_2t^2-a_3t^3+a_4t^4=0$$
and
$$b_0-b_1t+b_2t^2-b_3t^3+b_4t^4=0$$

be two quartics apolar to the three above quartics, then the determinants formed from the matrix of the coefficients of these two latter equations are proportional to the determinants in the first matrix; so that

$$\rho\Delta_{ij}=\Delta_{klm}{}^{*}$$

There is no loss in generality in placing $\rho=1$.

The condition that the quartic have a triple point is

$$\begin{vmatrix} a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{vmatrix}=0\dagger$$

that is,

$$\Delta_{01}\,.\,\Delta_{34}-\Delta_{02}\,.\,\Delta_{24}+\Delta_{03}\,.\,\Delta_{23}+\Delta_{12}\,.\,\Delta_{14}-\Delta_{13}{}^2+\Delta_{23}\,.\,\Delta_{12}=0$$

$$24\Delta_{234}\,.\,24\Delta_{012}-16\Delta_{134}\,.\,16\Delta_{013}+24\Delta_{124}\,.\,4\Delta_{014}+4\Delta_{034}\,.\,24\Delta_{023} \\ -(6\Delta_{024})^2+4\Delta_{014}.4\Delta_{034}=0$$

Substituting the values of the Δ_{ijk}'s, this becomes

$$\alpha^2[\beta\gamma(\alpha-\beta+2)\,(\beta-\gamma+2\beta\gamma)-(\alpha+\beta)(\beta+\gamma)\,(\alpha-\beta+2)\,(\beta-\gamma+2\beta\gamma) \\ +\beta(\alpha-\beta+2)\{\beta\gamma^2(\alpha-\beta+2)+(\beta+\gamma)\,(\beta-\gamma+2\alpha\beta)\} \\ +(\beta-\gamma+2\beta\gamma)\{(\alpha+\beta)\,(\alpha\gamma-\beta\gamma+2\alpha)+\,(\beta-\gamma+2\beta\gamma)\} \\ -\{\beta(\alpha\gamma-\beta\gamma+2\alpha+)(\beta-\gamma+2\beta\gamma)\}^2+\alpha\beta(\alpha-\beta+2)(\beta-\gamma+2\beta\gamma)]=0$$

Which reduces to

$$2\alpha^2(\beta-1)^2(\alpha-\beta)(\beta-\gamma)(\alpha-\gamma)=0$$

But it has been seen already that none of the conditions here obtained can be the necessary condition for a triple point.

A third proof of the non-existence of a triple point may be stated thus:

If possible, let the three cusped rational quintic have a triple point.

Subject the quintic to a quadratic transformation.

Let the vertices of its singular triangle be two of the cusps and the triple point, the cusps being at the vertices 1 and 2, and the triple point at 3.

The resulting curve is of order

$$10-3-2-2=3,\ddagger$$

---

* W. F. Meyer: "Apolaritat und Rationale Curven," Chap. i, p. 3.

† *Ibid.*, Chap. ii, p. 184.

‡ R. Sturm: "Die Lehre von dem Geometrischen Verwandtschaften," Vol. iv, p. 44.

having points of tangency on the sides $1'3'$ and $2'3'$ of the singular triangle in the transformed plane; also a branch of the curve goes through the vertex $3'$, since there is one extra intersection on the side 12 of the original triangle.

The original quintic had a third cusp, which in the transformation remains a cusp. Hence the new curve is a cuspidal cubic, and therefore of the third class. That is, from a point of the curve, but one tangent, excluding the one at the point itself, may be drawn.

But this curve would be on the vertex $3'$ and have as tangents the sides $1'3'$ and $2'3'$, which is clearly an impossibility. Accordingly, the three cusped rational quintic cannot have a triple point.

The Quadratic Transformation

If the quintic $x_i=\dfrac{t-\alpha_i}{(t-\beta_i)^2}$ $(i=1,\ 2,\ 3)$ be subjected to the quadratic transformation $x_i=\dfrac{1}{y_i}$ $(i=1,\ 2,\ 3)$, the resulting curve is $y_i=\dfrac{(t-\beta_i)^2}{t-\alpha_i}$

or

$$\begin{aligned} y_1&=(t-\beta_1)^2(t-\alpha_2)(t-\alpha_3)\\ y_2&=(t-\beta_2)^2(t-\alpha_3)(t-\alpha_1)\\ y_3&=(t-\beta_3)^2(t-\alpha_1)(t-\alpha_2). \end{aligned}$$

This is a curve of the fourth order. It passes through the vertices of the fundamental triangle of the transformed plane, and is at the same time tangent to the sides. That is, the fundamental triangle is simultaneously in-and-circumscribed to the quartic, its vertices being $t=\alpha_i$, and its points of tangency $t=\beta_i$.

It has been seen that the original quintic has three double points; and, further, that conditions can be imposed upon it so that it can acquire a fourth cusp or tacnode without degenerating. By the quadratic transformation, quintics of each of these types will go into quartics having three double points, a cusp or a tacnode respectively.

The existence, therefore, of one triangle in-and-circumscribed to the quartic led to the investigation of the number of such triangles in each of these cases.

It was shown that the rational quintic with three cusps could not have a triple point. Accordingly, the rational quartic with a triple point cannot have triangles in-and-circumscribed; for,

if it were possible, by subjecting the quartic to the quadratic transformation $y_i = \frac{1}{x_i}$ (with such a triangle as reference triangle), it would go into a quintic with a triple point. This has been shown impossible.

The Rational Quartic

Consider now the in-and-circumscribed triangles of the rational quartic.

Let its parametric equations be:

$$\begin{aligned} x_1 &= a_0t^4 + a_1t^3 + a_2t^2 + a_3t + a_4 \\ x_2 &= \phantom{a_0t^4 + {}} b_1t^3 + b_2t^2 + b_3t \\ x_3 &= \phantom{a_0t^4 + {}} c_1t^3 + c_2t^2 + c_3t \end{aligned}$$

thus making the vertex (1, 0, 0) a double point with parametric values thereat $t=0$ and $t=\infty$.

Let $(\alpha s)=0$ and $(\beta s)=0$ be the conditions upon a set of four parameters that they lie on a line.

Then
$$\begin{aligned} \alpha_0s_0 + \alpha_1s_1 + \alpha_2s_2 + \alpha_3s_3 + \alpha_4s_4 &= 0 \\ \beta_0s_0 + \beta_1s_1 + \beta_2s_2 + \beta_3s_3 + \beta_4s_4 &= 0, \end{aligned}$$

where $(\alpha\beta)_{ik}$ is proportional to the determinant Δ_{1mn} formed from the matrix of the coefficients of the parametric equations of the curve.*

That is,
$$\begin{aligned} |\alpha\beta|_{01} &\backsimeq 24\Delta_{234} = a_4\,|bc|_{23} = kX, \text{ say,} \\ |\alpha\beta|_{02} &\backsimeq 16\Delta_{134} = a_4\,|bc|_{13} = kY \\ |\alpha\beta|_{03} &\backsimeq 24\Delta_{124} = a_4\,|bc|_{12} = kZ \\ |\alpha\beta|_{04} &\backsimeq 96\Delta_{123} = \,|abc|_{123} = P \\ |\alpha\beta|_{14} &\backsimeq 24\Delta_{023} = a_0\,|bc|_{23} = X \\ |\alpha\beta|_{24} &\backsimeq 16\Delta_{013} = a_0\,|bc|_{13} = Y \\ |\alpha\beta|_{34} &\backsimeq 24\Delta_{012} = a_0\,|bc|_{12} = Z \\ |\alpha\beta|_{12} &= |\alpha\beta|_{13} = |\alpha\beta|_{23} = 4\Delta_{034} = 6\Delta_{024} = 4\Delta_{014} = 0 \end{aligned}$$

where $k = \frac{a_4}{a_0}$.

In $(\alpha s)=0$ and $(\beta s)=.0$ take s_i as the symmetric functions of $\lambda_1, \lambda_2, \lambda, \lambda$, so that,

$$\begin{aligned} s_0 = 1, \quad s_1 = \lambda_1 + \lambda_2 + 2\lambda, \quad s_2 = \lambda_1\lambda_2 + 2\lambda(\lambda_1+\lambda_2) + \lambda^2 \\ s_3 = 2\lambda_1\lambda_2\lambda + (\lambda_1, +\lambda_2)\lambda^2, \quad s_4 = \lambda_1\lambda_2\lambda^2 \end{aligned}$$

* W. F. Meyer: "Apolarität und Rationale Curven," Chap. i, p. 33.

Substituting these values for s_i,

$$(\alpha s)=[\alpha_0+\alpha_1(\lambda_1+\lambda_2)+\alpha_2\lambda_1\lambda_2]+2[\alpha_1+\alpha_2(\lambda_1+\lambda_2)+\alpha_3\lambda_1\lambda_2]\lambda \\ +[\alpha_2+\alpha_3(\lambda_1+\lambda_2)+\alpha_4\lambda_1\lambda_2]\lambda^2=0$$

$$(\beta s)=[\beta_0+\beta_1(\lambda_1+\lambda_2)+\beta_2\lambda_1\lambda_2]+2[\beta_1+\beta_2(\lambda_1+\lambda_2)+\beta_3\lambda_1\lambda_2]\lambda \\ +[\beta_2+\beta_3(\lambda_1+\lambda_2)+\beta_4\lambda_1\lambda_2]\lambda^2=0$$

Eliminate λ from these two equations:

$$E=\begin{vmatrix}\alpha_2+\alpha_3(\lambda_1+\lambda_2)+\alpha_4\lambda_1\lambda_2 & \alpha_0+\alpha_1(\lambda_1+\lambda_2)+\alpha_2\lambda_1\lambda_2 \\ \beta_2+\beta_3(\lambda_1+\lambda_2)+\beta_4\lambda_1\lambda_2 & \beta_0+\beta_1(\lambda_1+\lambda_2)+\beta_2\lambda_1\lambda_2\end{vmatrix}^2$$
$$-4\begin{vmatrix}\alpha_2+\alpha_3(\lambda_1+\lambda_2)+\alpha_4\lambda_1\lambda_2 & \alpha_1+\alpha_2(\lambda_1+\lambda_2)+\alpha_3\lambda_1\lambda_2 \\ \beta_2+\beta_3(\lambda_1+\lambda_2)+\beta_4\lambda_1\lambda_2 & \beta_1+\beta_2(\lambda_1+\lambda_2)+\beta_3\lambda_1\lambda_2\end{vmatrix}\times$$
$$\begin{vmatrix}\alpha_1+\alpha_2(\lambda_1+\lambda_2)+\alpha_3\lambda_1\lambda_2 & \alpha_0+\alpha_1(\lambda_1+\lambda_2)+\alpha_2\lambda_1\lambda_2 \\ \beta_1+\beta_2(\lambda_1+\lambda_2)+\beta_3\lambda_1\lambda_2 & \beta_0+\beta_1(\lambda_1+\lambda_2)+\beta_2\lambda_1\lambda_2\end{vmatrix}=0$$

Expanding the eliminant, and substituting therein for the $(\alpha\beta)_{ij}$'s their respective values, there results an equation in λ_1 and λ_2, which is the condition that the join of λ_1 and λ_2 be a tangent to the curve. By cyclically permuting, the conditions that the joins of λ_2 and λ_3, λ_1 and λ_3 be tangents to the curve can be written. These three equations are contained in the following *schema*, viz., equations (1), (2), (3), page 17.

From these equations there are derived three equations in S_1, S_2, S_3, the symmetric functions of λ_1, λ_2, λ_3. They are equations (4), (5), (6) of the *schema*, page 17.

It may be well to indicate here how these equations have been obtained. Multiply equation (1) by λ_3^4, (2) by λ_1^4, (3) by λ_2^4. Subtract these two by two, and throw out the factors $\lambda_3-\lambda_1$, $\lambda_1-\lambda_2$, $\lambda_2-\lambda_3$. This gives three new equations. Subtracting any two of these, and throwing out one of the above factors, equation (4) comes out. In an analogous way by multiplying the original equations by λ_1^3 and by λ_1^2, equations (5) and (6) are derived.

Equations (5) and (6) are linear in S_1 and S_2. From these the following values of S_1 and S_2 in terms of S_3 are derived:

$$S_1=\frac{\begin{gathered}-Y^4S_3^5+2Y^2(PX-3kYZ)S_3^4+2k(3PY^3+4X^2Y^2-2PXYZ \\ +4kXZ^3-8kY^2Z^2-X^3Z)S_3^3+2k^2(PXZ^2-kYZ^3-2PY^2Z \\ -5X^2YZ+12XY^3)S_3^2+k^3(kZ^4-4X^2Z^2-2PYZ^2+10XY^2Z \\ +4Y^4)S_3+2k^4Y^3Z\end{gathered}}{\begin{gathered}-X^2Y^2S_3^4+4k^2XY(XZ-2Y^2)S_3^3+k^2(16XY^2Z-4kXY^2Z \\ -3X^2Z^2-15Y^4)S_3^2+4k^3YZ(XZ-2Y^2)S_3-k^4Y^2Z^2\end{gathered}}$$

$$S_2=\frac{\begin{gathered}2XY^3S_3^5+(X^4+4kY^4-4kX^2Z^2-2PX^2Y+10kXY^2Z)S_3^4 \\ +2k(PX^2Z-2PXY^2-X^3Y-5kXYZ^2+12kY^3Z)S_3^3 \\ +2k^2(4kY^2Z^2+3PY^3-4PXYZ+4X^3Z-8X^2Y^2-kXZ^3)S_3^2 \\ +2k^3Y^2(PZ-3XY)S_3-k^4Y^4\end{gathered}}{\begin{gathered}-X^2Y^2S_3^4+4k^2XY(XZ-2Y^2)S_3^3+k^2(16XY^2Z-4kXY^2Z \\ -3X^2Z^2-15Y^4)S_3^2+4k^3YZ(XZ-2Y^2)S_3-k^4Y^2Z^2\end{gathered}}$$

TABLE I

Y^2	$2XY$	$2PY$ $-4kZ^2$	X^2	$2PX$ $-6kYZ$	P^2 $+2kY^2$ $-8kXZ$	$2kXZ$ $-4kY^2$	$2kPZ$ $-6kXY$	$2kPy$ $-4kX^2$	k^2Z^2	$2k^2YZ$	k^2Y^2	
$\lambda_1{}^4\lambda_2{}^4$	$\lambda_1{}^3\lambda_2{}^3(\lambda_1+\lambda_2)$	$\lambda_1{}^3\lambda_2{}^3$	$\lambda_1{}^2\lambda_2{}^2(\lambda_1+\lambda_2)^2$	$\lambda_1{}^2\lambda_2{}^2(\lambda_1+\lambda_2)$	$\lambda_1{}^2\lambda_2{}^2$	$\lambda_1\lambda_2(\lambda_1+\lambda_2)^2$	$\lambda_1\lambda_2(\lambda_1+\lambda_2)$	$\lambda_1\lambda_2$	$(\lambda_1+\lambda_2)^2$	$\lambda_1+\lambda_2$	1	(1)
$\lambda_2{}^4\lambda_3{}^4$	$\lambda_2{}^3\lambda_3{}^3(\lambda_2+\lambda_3)$	$\lambda_2{}^3\lambda_3{}^3$	$\lambda_2{}^2\lambda_3{}^2(\lambda_2+\lambda_3)^2$	$\lambda_2{}^2\lambda_3{}^2(\lambda_2+\lambda_3)$	$\lambda_2{}^2\lambda_3{}^2$	$\lambda_2\lambda_3(\lambda_2+\lambda_3)^2$	$\lambda_2\lambda_3(\lambda_2+\lambda_3)$	$\lambda_2\lambda_3$	$(\lambda_2+\lambda_3)^2$	$\lambda_2+\lambda_3$	1	(2)
$\lambda_3{}^4\lambda_1{}^4$	$\lambda_3{}^3\lambda_1{}^3(\lambda_3+\lambda_1)$	$\lambda_3{}^3\lambda_1{}^3$	$\lambda_3{}^2\lambda_1{}^2(\lambda_3+\lambda_1)^2$	$\lambda_3{}^2\lambda_1{}^2(\lambda_3+\lambda_1)$	$\lambda_3{}^2\lambda_1{}^2$	$\lambda_3\lambda_1(\lambda_3+\lambda_1)^2$	$\lambda_3\lambda_1(\lambda_3+\lambda_1)$	$\lambda_3\lambda_1$	$(\lambda_3+\lambda_1)^2$	$\lambda_3+\lambda_1$	1	(3)
0	$S_3{}^3$	0	$S_3{}^2S_2$	0	$-S_3{}^2$	$-S_3{}^2$	$-S_2S_3$	$-S_1S_3$	$-S_2{}^2$	$S_3-S_1S_2$	$S_2-S_1{}^2$	(4)
$S_3{}^3$	0	0	$-S_1S_3{}^2$	$-S_3{}^2$	0	$-S_2S_3$	0	S_3	S_3	S_2	S_1	(5)
$S_2S_3{}^2$	$S_1S_3{}^2$	$S_3{}^2$	$S_3{}^2$	0	0	$-S_1S_3$	$-S_3$	0	$-S_2$	0	1	(6)

These values substituted in equation (4) give an equation of the eleventh degree in S_3. The constant term and the coefficients of S_3^{11} and S_3 vanish identically, thus reducing the equation, after dividing through by S_3^2, to the eighth degree.

The coefficients of this octavic are such that when $Y=0$, the coefficients of S_3^8, S_3^7, and S_3, as well as the constant term, become identically zero, and the equation then reduces to a quartic. This condition, viz., $Y=0$, is the condition, as will be seen later, that the quartic have a tacnode.

The equation has eight possible solutions, each of which gives a single value for S_1 and S_2. There are then eight sets of values for S_1, S_2 and S_3, each set leading to an in-and-circumscribed triangle, the vertices of which are found from the equation

$$\lambda^3-S_1\lambda^2+S_2\lambda-S_3=0.$$

It is to be noted that the number of triangles given here is the same as that found by Cayley's formula (page 4) for this case.

The Quartic With a Tacnode

Let the parametric equations of the quartic be

$$\begin{aligned} x_1 &= a_0t^4+a_1t^3+a_2t^2+a_3t+a_4 \\ x_2 &= \qquad\quad b_1t^3+b_2t^2+b_3t \\ x_3 &= \qquad\qquad\quad c_2t^2 \end{aligned}$$

thus making the vertex (1, 0, 0) a tacnodal point with parametric values thereat $t=0$ and $t=\infty$, and at the same time making $x_3=0$ a tangent to the curve at the same point.

Let $(\alpha s)=0$ and $(\beta s)=0$ be the "Schnittpunktformen," that is, the conditions that a set of four points lie on a line. Then using the notation of the previous case,

$$\begin{aligned} &|\alpha\beta|_{01} \backsimeq 24\Delta_{234} = -a_4b_3c_2 = kX, \text{ say,} \\ &|\alpha\beta|_{03} \backsimeq 24\Delta_{124} = a_4b_1c_2 = kZ \\ &|\alpha\beta|_{04} \backsimeq 96\Delta_{123} = -c_2\,|ab|_{13} = P \\ &|\alpha\beta|_{14} \backsimeq 24\Delta_{023} = -a_0b_3c_2 = X \\ &|\alpha\beta|_{34} \backsimeq 24\Delta_{012} = a_0b_1c_2 = Z \\ |\alpha\beta|_{12} = {} &|\alpha\beta|_{13} = |\alpha\beta|_{23} = 4\Delta_{034} = 6\Delta_{024} = 4\Delta_{014} = 0 \\ &|\alpha\beta|_{02} = |\alpha\beta|_{24} = Y = 0 \end{aligned}$$

Substituting $Y=0$ in equations (4), (5), (6), page 17, the following relations between S_1, S_2, S_3 result:

$$X^2S_3{}^2S_2-(P^2-6kXZ)S_3{}^2-2kPZS_2S_3+4kX^2S_1S_3-k^2Z^2S_2{}^2=0 \quad\text{. . I.}$$

$$-X^2S_3{}^2S_1-2PXS_3{}^2-2kXZS_2S_3-4kX^2S_3+k^2Z^2S_3=0 \quad\text{. . . . II.}$$

$$(X^2-4kZ^2)S_3-2kXZS_1S_3-2kPZS_3-k^2Z^2S_2=0 \quad\text{. . . . . . III.}$$

From II and III, linear in S_1 and S_2, it follows that:

$$S_1=\frac{2X(4kZ^2-X^2)S_3{}^2+2kPXZS_3-kZ(4kX^2-k^2Z^2)}{-3kX^2ZS_3} \quad\text{. . . IV.}$$

$$S_2=\frac{-X(4kZ^2-X^2)S_3{}^2+2kPXZS_3+2kZ(4kX^2-k^2Z^2)}{-3k^2XZ^2} \quad\text{. . . V.}$$

Substituting these values for S_1 and S_2 in I, there results the following equation in S_3:

$$\begin{aligned} &4X^2(4kZ^2-X^2)(X^2-kZ^2)S_3{}^4-4kPX^2Z(X^2+2kZ^2)S_3{}^3 \\ &\quad -k^2XZ(16X^4-32k^2X^2Z^2+P^2XZ+16k^2Z^4)S_3{}^2 \\ &\quad -4k^3PXZ^2(2X^2+kZ^2)S_3-4k^4Z^2(4X^2-kZ^2)(X^2-kZ^2)=0. \end{aligned}$$

This equation factors into:

$$[2X(X+2\sqrt{kZ})(X+\sqrt{kZ})S_3^2+kPXZS_3+2k^2Z(2X+\sqrt{kZ})(X+\sqrt{kZ})]$$
$$[2X(X-2\sqrt{kZ})(X-\sqrt{kZ})S_3^2+kPXZS_3+2k^2Z(2X-\sqrt{kZ})(X-\sqrt{kZ})]=0 \quad \ldots\ldots \text{VI.}$$

There are four possible solutions for S_3. Each value gives a single value for S_1 and S_2. Hence, there are four sets of values for S_i, each set determining an in-and-circumscribed triangle.

The fact that equation VI. factors into two quadratic factors must be especially noticed. This indicates that the triangles fall into two sets of two each, and not into one irreducible set of four triangles.

The Quartic with a Triple Point

Let $(\alpha s)=0$ and $(\beta s)=0$ be the "Schnittpunktformen" apolar to the three equations representing parametrically the rational quartic, viz.,

$$\begin{aligned} x_1 &= a_0t^4+a_1t^3+a_2t^2+a^3t+a_4 \\ x_2 &= \quad b_1t^3+b_2t^2+b_3t \\ x_3 &= \quad c_1t^3+c_2t^2+c_3t \end{aligned}$$

Then the condition that the quartic have a triple point is

$$\begin{vmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_0 & \beta_1 & \beta_2 & \beta_3 \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \end{vmatrix} = 0$$

which reduces to $|\alpha\beta|_{01}\cdot|\alpha\beta|_{34}-|\alpha\beta|_{02}\cdot|\alpha\beta|_{24}=0$.

That is, $|\alpha\beta|_{01}:|\alpha\beta|_{02}=|\alpha\beta|_{24}:|\alpha\beta|_{34}$, which in the notation previously used is $kX:kY=Y:Z$, or $XZ=Y^2$.

It is possible, therefore, to take $X=1$, $Y=m$, $Z=m^2$, and also $k=1$.

Substituting these values in equations (4), (5), (6), page 17, they reduce to:

$$2mS_3^3-(P^2-8m^2)S_3^2+S_3^2S_2+(6m-2Pm^2)S_2S_3+(4-2mP)S_1S_3$$
$$-m^4S_2^2+(S_3-S_1S_2)2m^3+m^2(S_2-S_1^2)=0 \quad \ldots\ldots \text{I.}$$

$$m^2S_3^3-(2P-6m^3)S_3^2-S_1S_3^2+(m^4+2mP-4)S_3+2m^2S_2S_3$$
$$+2m^3S_3+m^2S_1=0 \quad \ldots\ldots \text{II.}$$

$$m^2S_2S_3^2+2mS_1S_3^2+(1+2mP-4m^4)S_3^2+2m^2S_1S_3$$
$$-(2Pm^2-6m)S_3+m^4S_2+m^2=0 \quad \ldots\ldots \text{III.}$$

From equations II and III, linear in S_1 and S_2, it follows that:

$$S_1=\frac{m^2S_3^3+(4m^3-2P)S_3^2-(m^4-2Pm+6)S_3-2m}{(S_3+m)^2}$$

$$S_2=\frac{-2m^3S_3^3-(6m^4-2Pm+1)S_3^2-(2Pm^2-4m)S_3+m^2}{m^2(S_3+m)^2}$$

Substituting these values in I, this equation becomes:

$$(-m^8+2Pm^5-2m^4-P^2m^2+2Pm-1)S_3^2(S_3+m)^4=0.$$

Neither one of the values $S_3=0,\ -m$ gives proper triangles.

The Quartic with a Cusp

Let the parametric equations be:

$$x_1=(t^2-a^2)^2=t^4-2a^2t^2+a^4$$
$$x_2=t^2(t-b)^2=t^4-2bt^3+b^2t^2$$
$$x_3=t^2(t-c)^2=t^4-2ct^3+c^2t^2$$

thus placing the cusp at the vertex (1, 0, 0) with $t=0$ thereat, and also making the sides of the triangle of reference tangents to the curve. The sides 13 and 12 have points of tangency at $t=b$ and $t=c$ respectively; while 23 is a double tangent with $t=\pm a$ at the points of tangency.

Let $(\alpha s)=0$ and $(\beta s)=0$ be the conditions that a set of four points lie on a line, then,

$$|\alpha\beta|_{34}\approx 24\Delta_{012}=2(2a^2-bc)(c-b)$$
$$|\alpha\beta|_{23}\approx 4\Delta_{014}=-2a^4(c-b)$$
$$|\alpha\beta|_{13}\approx 6\Delta_{024}=a^4(c^2-b^2)$$
$$|\alpha\beta|_{03}\approx 24\Delta_{124}=-2a^4bc(c-b)$$

$$|\alpha\beta|_{01}=|\alpha\beta|_{02}=|\alpha\beta|_{04}=|\alpha\beta|_{12}=|\alpha\beta|_{14}=|\alpha\beta|_{24}=0$$

In $(\alpha s)=0$, $(\beta s)=0$ let s_i be the symmetric functions of λ_1, λ_2, λ, λ. Substitute for s_i their respective values. From these equations eliminate λ. The result is the condition that the join of λ_1 and λ_2 be a tangent to the curve. Upon substituting for $\alpha\beta_{ik}$ their respective values the eliminant reduces to, after throwing out the linear factor, $2\lambda_1\lambda_2-(b+c)(\lambda_1+\lambda_2)+2bc$, and thus making it a $(3-3)$ correspondence:

$$a^4(b+c)(\lambda_1+\lambda_2)^3+6a^4(\lambda_1+\lambda_2)^2\lambda_1\lambda_2-2a^4bc(\lambda_1+\lambda_2)^2$$
$$-8(2a^2-bc)\lambda_1^3\lambda_2^3-4a^4(b+c)(\lambda_1+\lambda_2)\lambda_1\lambda_2-8a^4\lambda_1^2\lambda_2^2=0$$

The linear factor thrown out is the condition that the join of λ_1 and λ_2 pass through the cusp. The condition imposed upon

this join is that the remaining two parameters shall be equal, a condition which is satisfied if the line passes through the cusp.

Two similar equations, giving the conditions that the joins of λ_1 and λ_2, λ_2 and λ_3 be tangents to the curve, can be derived. From these three, then, the following equations in S_1, S_2, and S_3, the symmetric functions of λ_1, λ_2, and λ_3 can be derived. They are:

$$8(2a^2-bc)S_2S_3-a^4(b+c)(S_2-S_1^2)+2a^4S_3-2a^4bcS_1=0 \ldots \quad \text{I.}$$

$$(b+c)(S_2^2-S_1S_3)+6S_2S_3+2bcS_3=0 \ldots\ldots\ldots\ldots \quad \text{II.}$$

$$S_3^2+(b+c)S_2S_3+bcS_2^2=0 \ldots\ldots\ldots\ldots\ldots \quad \text{III.}$$

From III, $S_3=-bS_2$ or $-cS_2$

Substituting $S_3=-bS_2$ in II, gives $S_1=\dfrac{(5b-c)S_2+2b^2c}{b(b+c)}$,

and substituting these values in I, there results

$$S_2=0, \text{ or } \frac{a^4b^2\{(b+c)(3b+c)-2c(5b-c)\}}{a^4(5b-c)^2-8b^3(2a^2-bc)(b+c)}$$

Similarly using $S_3=-cS_2$, gives

$$S_1=\frac{(5c-b)S_2+2bc^2}{c(b+c)}$$

and
$$S_2=\frac{a^4c^2\{(b+c)(b+3c)-2b(5c-b)\}}{a^4(5c-b)^2-8c^3(2a^2-bc)(b+c)}, \text{ or } 0.$$

The value $S_2=0$ evidently does not lead to a solution. Hence, there are two possible solutions, one for each value of S_3.

It is to be noted that the number of triangles given here is the same as that found by Cayley's formula (page 4) for this case.

The (3−3) *Correspondence*

The in-and-circumscribed triangles of the cuspidal quartic were found by means of a (3−3) correspondence. This correspondence is not, however, of the most general kind. The most general correspondence of this type is that set up by means of the lines drawn from a point of a conic to a line-cubic.

The question then arises, what are the conditions which the line-cubic must satisfy, either in itself or in its relations to the conic, in order that the general (3−3) correspondence reduce to the type set up by the method here employed?

Take as the conic the norm-conic:

$$x_0 = 1$$
$$x_1 = 2\lambda$$
$$x_2 = \lambda^2$$

Let the line-cubic be $(a\xi)^3 = 0$.

The condition that the join of two points λ_1, λ_2, of the conic be a line of the cubic is:

$$\xi_i = \left\| \begin{matrix} 1 & 2\lambda_1 & \lambda_1^2 \\ 1 & 2\lambda_2 & \lambda_2^2 \end{matrix} \right\| = \sigma_2 : -\frac{\sigma_1}{2} : 1,$$

where σ_1, σ_2 are the symmetric functions of λ_1 and λ_2.

Substitute these values in $(a\xi)^3 = 0$, thus giving the equation of the $(3-3)$ correspondence between λ_1 and λ_2:

$$a_{000}\sigma_2{}^3 - a_{111}\frac{\sigma_1{}^3}{8} + a_{222} - 3a_{001}\frac{\sigma_1\sigma_2{}^2}{2} + 3a_{002}\sigma_2{}^2 + 3a_{011}\frac{\sigma_1{}^2\sigma_2}{4} + 3a_{112}\frac{\sigma_1{}^2}{4}$$
$$3a_{022}\sigma_2 - \frac{3}{2}a_{122}\sigma_1 - 3a_{012}\sigma_1\sigma_2 = 0$$

These coefficients being all independent, this is the most general $(3-3)$ correspondence. The equation of the $(3-3)$ correspondence set up by the rational cuspidal quartic has but six terms, lacking the constant term and those in $\sigma_1\sigma_2{}^2$, σ_2, σ_1. In order that the general correspondence become of the same type as the special one, then

$$a_{001} = a_{022} = a_{122} = a_{222} = 0$$

The equation of the line cubic then reduces to:

$$a_{000}\xi_0{}^3 + 3a_{002}\xi_0{}^2\xi_2 + 3a_{011}\xi_0\xi_1{}^2 + 6a_{012}\xi_0\xi_1\xi_2 + 3a_{112}\xi_1{}^2\xi_2 + a_{111}\xi_1{}^3 = 0$$

This cubic lacks the terms in $\xi_2{}^3$ and $\xi_2{}^2$, thus indicating that the line $x_2 = 0$ is a double tangent of the line-cubic, and is at the same time a line of the conic.

The results obtained indicate some of the conditions which must be imposed upon the line cubic so that the general $(3-3)$ correspondence may reduce to the special type here under consideration. It is to be noted, at the same time, that these conditions are necessary but may not be sufficient.

R. A. Roberts, in a paper entitled, "On Polygons Circumscribed about a Conic and Inscribed in a Cubic," proposes "to consider . . . the general problem of finding conics and cubics related to each other in such a manner that it may be possible to circum-

scribe about the conic an infinite number of polygons which are inscribed in the cubic."*

That is, he takes the lines of a conic and the points of a cubic, which is the dual of what has been used above, and finds various relations between the two curves so that the results obtained are always poristic.

It has been shown here that, if a tangent to a point conic is a double line of a line cubic, the solution cannot be poristic. Dually, this would say that if a point cubic have a double point and this double point be on the line conic, the solution cannot be poristic.

The $(4-4)$ *Correspondence*

The correspondence set up in the case of the quartic with three double points is a $(4-4)$ correspondence. As in the preceding case, it is, however, not of the most general type. The most general $(4-4)$ correspondence is set up by means of a point-conic and a line-quartic. As in the preceding instance, what are the conditions which the quartic curve must satisfy, either in itself or in its relations to the conic, so that the general $(4-4)$ correspondence reduce to the type set up by the method employed here?

Let the conic be the norm conic: $x_0=1$; $x_1=2\lambda$; $x_2=\lambda^2$.

And let the line quartic be $(a\xi)^4=0$.

Then the conditions that the joins of two points λ_1 and λ_2 of the conic be a line of the quartic is:

$$\xi_i=\left\|\begin{matrix}1 & 2\lambda_1 & \lambda_1^2\\ 1 & 2\lambda_2 & \lambda_2^2\end{matrix}\right\|=\sigma_2:-\frac{\sigma_1}{2}:1$$

where σ_1, σ_2 are the symmetric functions of λ_1 and λ_2.

Substitute these values in $(a\xi)^4=0$, thus giving the equation of the $(4-4)$ correspondence between λ_1 and λ_2:

$$a_{0000}\sigma_2{}^4-4a_{0001}\frac{\sigma_2{}^3\sigma_1}{2}+4a_{0002}\sigma_2{}^3+6a_{0011}\frac{\sigma_2{}^2\sigma_1{}^2}{4}-12a_{0012}\frac{\sigma_2{}^2\sigma_1}{2}+6a_{0022}\sigma_2{}^2$$
$$-4a_{0111}\frac{\sigma_2\sigma_1{}^3}{8}+12a_{0112}\frac{\sigma_2\sigma_1{}^2}{4}-12a_{0122}\frac{\sigma_2\sigma_1}{2}+4a_{0222}\sigma_2+a_{1111}\frac{\sigma_1{}^4}{16}-4a_{1112}\frac{\sigma_1{}^3}{8}$$
$$+6a_{1122}\frac{\sigma_1{}^2}{4}-4a_{1222}\frac{\sigma_1}{2}+a_{2222}=0$$

These coefficients are all independent, and so this is the most general type of $(4-4)$ correspondence.

---

* *Proceedings of the London Mathematical Society*, Vol. xvii, p. 158.

The equation of the $(4-4)$ correspondence set up by the rational quartic has but twelve terms, lacking those in $\sigma_1{}^4$, $\sigma_1{}^3\sigma_2$, $\sigma_1{}^3$. (See Eqn. 1, p. 17). In order that the general correspondence become of the same type as this special one, then,

$$a_{1111}=a_{0111}=a_{1112}=0.$$

The equation of the line-quartic then reduces to:

$$a_{0000}\xi_0{}^4+4a_{0001}\xi_0{}^3\xi_1+4a_{0002}\xi_0{}^3\xi_2+6a_{0011}\xi_0{}^2\xi_1{}^2+12a_{0012}\xi_0{}^2\xi_1\xi_2+6a_{0022}\xi_0{}^2\xi_2{}^2$$
$$+12a_{0112}\xi_0\xi_1{}^2\xi_2+12a_{0122}\xi_0\xi_1\xi_2{}^2+4a_{0222}\xi_0\xi_2{}^3+6a_{1122}\xi_1{}^2\xi_2{}^2+4a_{1222}\xi_1\xi_2{}^3$$
$$+a_{2222}\xi_2{}^4=0$$

This quartic lacks the terms in $\xi_1{}^4$ and $\xi_1{}^3$, thus indicating that the line $x_1=0$ is a double line of the quartic. This line is the join of the points of the norm-conic whose parameters are $\lambda=0$, $\lambda=\infty$, which are also the parameters at one of the double points of the rational quartic. Consequently, the lines, which are the joins of the points whose parameters are the same as the parameters at the other two double points, are also double lines of the line-quartic. These conditions are some of those which the quartic must satisfy, but are not necessarily all.

In the case of the quartic with a tacnode, the $(4-4)$ correspondence has only eight terms (as is readily seen by placing $Y=0$ in Eqn. 1, p. 17). Then the general correspondence, in order to be of the same type, must have, in addition to

$$a_{1111}=a_{0111}=a_{1112}=0,$$

also
$$a_{0000}=a_{0001}=a_{1222}=a_{2222}=0.$$

The line quartic then becomes,

$$4a_{0002}\xi_0{}^3\xi_2+6a_{0011}\xi_0{}^2\xi_1{}^2+12a_{0012}\xi_0{}^2\xi_1\xi_2+6a_{0022}\xi_0{}^2\xi_2{}^2+12a_{0112}\xi_0\xi_1{}^2\xi_2$$
$$+12a_{0122}\xi_0\xi_1\xi_2{}^2+4a_{0222}\xi_0\xi_2{}^3+6a_{1122}\xi_1{}^2\xi_2{}^2=0.$$

The absence of the terms in $\xi_1{}^4$ and $\xi_1{}^3$ indicates that the line $x_1=0$ is a double line of the quartic; while the absence of the terms $\xi_0{}^4$ and $\xi_2{}^4$ indicates that the lines $x_0=0$ and $x_2=0$ are also lines of the quartic. The absence of the terms in $\xi_0{}^3\xi_1$ and $\xi_1\xi_2{}^3$ means that the quartic passes through the vertices 0 and 2, and since the lines $x_0=0$ and $x_2=0$ are lines of the quartic these vertices must be points of tangency.

It was shown, in the case of the quartic with three double-points, that the lines joining the points of the conic, whose parameters are the same as the parameters at the double-points, are lines of the quartic, so in the case of the tacnodal quartic, where

two of the three double-points come together at the tacnode, the line joining the points of the conic whose parameters are the same as those of the remaining double-point, is a line of the quartic.

Reality of Solutions

A solution is to be regarded as real if the cubic giving the vertices, viz.,

$$t^3-S_1t^2+S_2t-S_3=0,$$

has real coefficients. In all the cases examined, the equations obtained enable S_1 and S_2 to be expressed rationally in terms of S_3. Every real value of S_3 will, therefore, lead to real values of S_1 and S_2, and, accordingly, to a real solution. The number of real solutions will therefore be simply the number of real roots of the equation in S_3. It may be noted that the above cubic will not always lead to a triangle with three real vertices. In the tacnodal case, the special values given (p. 30) yield four real solutions. Of these four real values of S_3, only one leads to a triangle with three real vertices.

Special Cases

The following section is added in order to include a few cases showing in-and-circumscribed triangles with three real vertices. It is to be noted that in the various cases it may not be possible to draw the maximum number of real triangles. For the rational quartic, a case was found where three of the possible eight triangles could be drawn. For the quartic with a tacnode only one could be found. However, in the case of the cuspidal quartic, it was possible to find two triangles with three real vertices each.

(*a*) *The Rational Quartic*

It is proposed, here, to find a rational quartic symmetrical as to the fundamental triangle of reference.

Let the three double points be at the vertices of the triangle, and have as parameters thereat $(0, \infty)$, $(1, -\frac{m^2+m+1}{m})$, $(-m, m^2+m+1)$. These values are determined as follows: If the six parameters taken in order of continuity along the path of the curve be $0, 1, a, \infty, b, -m$, then a and b can be determined in terms of m by means of the relations,

$$(0, 1/a, \infty)=(a, \infty/b, -m)=(b, -m/0, 1),$$

whence $a=m^2+m+1$ and $b=-\frac{m^2+m+1}{m}$,

Then

$$x_1=k_1t(t-1)(t+\frac{m^2+m+1}{m})$$

$$x_2=k_2t(t+m)(t-[m^2+m+1])$$

$$x_3=(t-1)(t+m)(t+\frac{m^2+m+1}{m})(t-[m^2+m+1])$$

It is necessary to determine k_1, k_2 so that the curve have triangular symmetry. This will be the case if the tangents at the double points be oppositely inclined.

The tangent at 0 is, $m^2\frac{x_1}{k_1}-\frac{x_2}{k_2}=0$

The tangent at ∞ is, $\frac{x_1}{k_1}-\frac{x_2}{k_2}=0$.

These are oppositely inclined if $k_1{}^2=m^2k_2{}^2$. Take $k_1=mk$; $k_2=-k$.

Substituting these values in the above equations, the equations of the tangents at 1 and $-\dfrac{m^2+m+1}{m}$ are found to be:

$$x_1(1+m)^2+kx_3=0 \text{ and } x_1(1+m)^2+km^2x_3=0.$$

These are oppositely inclined if $k^2=\dfrac{(1+m)^4}{m^2}$. Take $k=\dfrac{(1+m)^2}{m}$.

The parametric equations of the curve then become

$$x_1=m(1+m)^2t(t-1)(t+\frac{m^2+m+1}{m})$$
$$x_2=-(1+m)^2t(t+m)(t-[m^2+m+1])$$
$$x_3=m(t-1)(t+m)(t+\frac{m^2+m+1}{m})(t-[m^2+m+1])$$

It may readily be shown that the tangents at the remaining double point are oppositely inclined, and that, consequently, the curve has triangular symmetry.

The line $x_1=x_2$ intersects the curve, aside from $t=0$, ∞, in the parameter-pair

$$t^2-(m^2+m+1)=0$$

It is required to find what value m must have so that the point $t=\sqrt{m^2+m+1}$ may be a vertex of an in-and-circumscribed triangle. On account of the triangular symmetry, it is clear that the other two vertices will be the remaining intersections of the tangents at $t=-\sqrt{m^2+m+1}$ with the curve, and that there will be three such triangles.

The equation of the tangent at this point can be derived. From this the symmetric functions of the parameters at the three vertices, in terms of m, can be written down. Substituting these in equation 5, p. 17, this reduces to

$$69m^{12}+312m^{11}+508m^{10}+422m^9-2564m^8-6238m^7-8736m^6$$
$$-7160m^5-3551m^4-304m^3+668m^2+396m+64=0,$$

a positive root of which is $m=2.15$.

The parametric equations of the curve become

$$x_1=2.15(3.15)^2t(t-1)(t+3.615)$$
$$x_2=-(3.15)^2t(t+2.15)(t-7.7725)$$
$$x_3=2.15(t-1)(t+2.15)(t+3.615)(t-7.7725).$$

The vertices of one triangle, given by $t=\sqrt{m^2+m+1}$ and equation of parameters giving the intersections of the tangent at $t=-\sqrt{m^2+m+1}$ with the curve, are $t=2.788$, 1.09, and 6.898.

On account of the symmetry of the curve, the vertices of the other two triangles may be found in an analogous way. For the construction of these triangles see Figure 1.

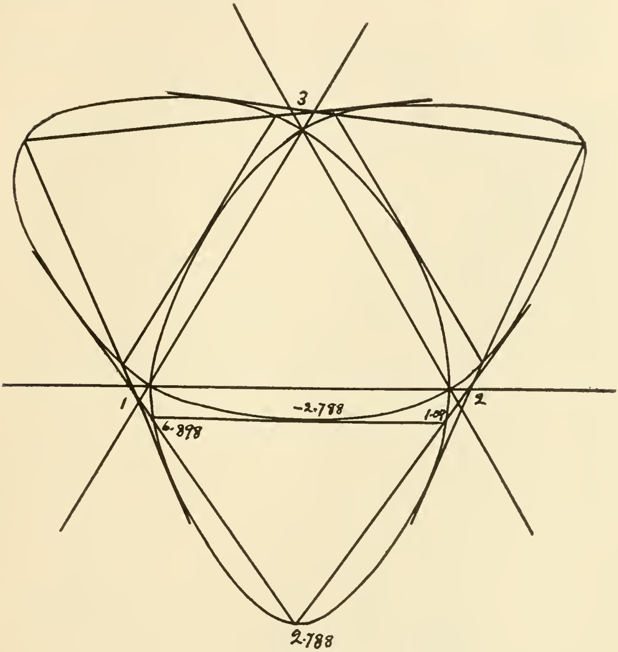

FIGURE 1.

(b) The Quartic with a Tacnode

Let the parametric equations be

$$x_1 = \left(t - \frac{4}{3}\right)^2 (t-3)^2$$
$$x_2 = t(t-4)\left(t - \frac{4}{3}\right)$$
$$x_3 = 5t^2$$

thus making the vertex (1, 0, 0) a tacnodal point with parametric

values thereat $t=0$, ∞, and at the same time making $x_3=0$ a tangent to the curve at the same point, and also making $x_1=0$ a double tangent with parametric values $t=3$ and $t=\frac{4}{3}$ at the points of tangency.

From the matrix of the coefficients,

$$X=-\frac{80}{3},\quad Z=5,\quad P=\frac{520}{9},\quad k=16.$$

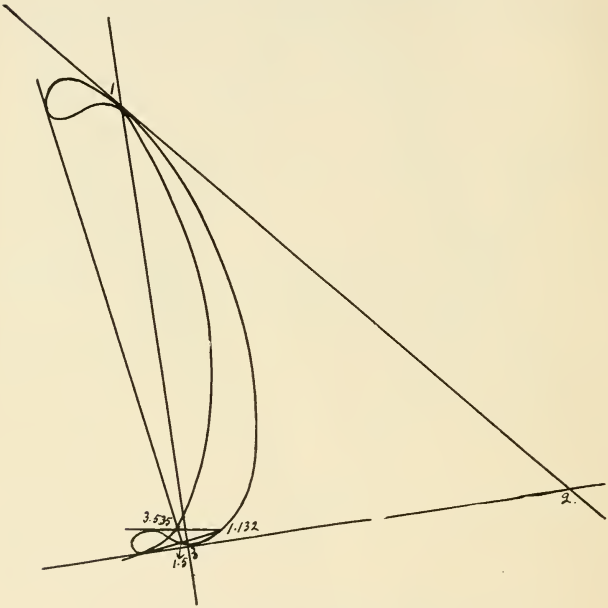

FIGURE 2.

Substituting these values in equation VI, p. 20, there follows:

$$[S_3^2-26S_3+120]\,[-35S_3^2-26S_3+2\times 1\;\;1\times 3\times 28]=0,$$

whence $S_3=6$, 20, 6.9, or -7.65.

$S_3=6$ is the only value which leads to triangle with three real vertices. Substituting this value in equations IV and V (p. 19), then

$$S_1=\frac{37}{6}\qquad\text{and}\qquad S_2=11.$$

The vertices of the triangle are then given by the roots of the equation:

$$t^3 - \frac{37}{6}t^2 + 11t - 6 = 0$$

$$(t-1.5)(t-3.535)(t-1.132) = 0$$

For the construction of this triangle, see Figure 2.

(*c*) *The Quartic with a Cusp*

The following values for a, b, c, lead to two solutions, both of which give real triangles, as is seen from the accompanying table:

a	b	c	S_1	S_2	S_3	t_1	t_2	t_3
1	−4	2	8.487	−.177	−.708	.283	8.496	−.292
			7	.293	−.586	6.94	.319	−.264

For the construction of these triangles, see Figure 3.

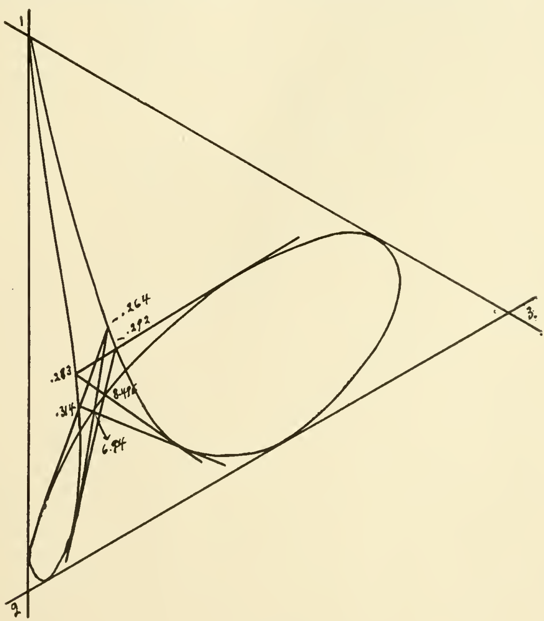

FIGURE 3.

BIOGRAPHICAL SKETCH

Joseph Nelson Rice was born at Weymouth, Nova Scotia, on the 26th of December, 1890. He received his elementary and high school education at the public school of this town. In the fall of 1906 he entered St. Francis Xavier's College, Antigonish, N. S., and was graduated therefrom with the degree of Bachelor of Arts in 1910. In 1912 he received the degree of Master of Arts. During the years 1910 to 1913 he was an instructor in the department of Mathematics at this same college. In the fall of 1913, he entered the Catholic University of America as a graduate student in the department of Mathematics. He has followed courses under Dr. Landry, Professor of Mathematics; Dr. Shea, Professor of Physics; and Mr. Crook, Instructor in Mechanics.

He desires to take this opportunity of expressing his thanks to Professor Landry for many valuable suggestions offered during the preparation of this thesis.